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New exact solutions for a free boundary system

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Abstract. A method for constructing explicit exact solutions to nonlinear evolution equations is further developed. The method is based on consideration of a fixed nonlinear partial differential equation together with an additional generating condition in the form of a linear high-order ordinary differential equation. The method is then applied to a free boundary problem based on the process of precipitant-assisted protein crystal growth.

1. Introduction

In [1, 2] a one-dimensional model for precipitant-assisted protein crystal growth is discussed. The partial differential equations (PDEs) for this model have the form

$$u_t = (D_1 u_x)_x + \mu u_x \qquad v_t = (D_2 v_x)_x + \mu v_x \tag{1}$$

where for $x, t > 0$ the functions $u = u(t, x)$ and $v = v(t, x)$ are the protein and precipitant concentrations in a liquid phase, respectively, $\mu = \mu(t)$ is the velocity of a free interface at $x = 0$, and subscripts t and x denote differentiation with respect to these variables. The diffusivities $D_1 = D_1(u, v)$ and $D_2 = D_2(u, v)$ are assumed to depend explicitly on the concentrations u and v, so system (1) is nonlinear. These two equations are coupled together at the free interface by the following boundary conditions:

$$\mu(u_s - u) = D_1 u_x \qquad \mu(v_s - v) = D_2 v_x \qquad f(u, v) = 0 \tag{2}$$

where u_s and v_s are the protein and precipitant concentrations in a solid phase ($x < 0$). The solid-phase concentrations may implicitly depend on time because of the moving interface, though often they are constant. The first two equations of (2) are classical Stefan conditions representing mass conservation at the interface; the third, $f = 0$, is a solubility relation that guarantees the two liquid-phase constituents are in equilibrium at the interface. In addition to (1) and (2), one would need far-boundary and initial conditions to determine uniquely the solution (u, v, μ).

It is known that construction of an exact solution for a nonlinear boundary value problem (BVP) can be very difficult. Certain exact solutions for Stefan-like problems are well known (see, e.g., [3–6] and their references, and the more recent work in [7]), but generally only for constant diffusivities. One of the present authors has previously found exact solutions

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for certain nonlinear BVPs associated with heat diffusion [8, 9] (see also the references in [10]).

In the present paper, we construct wide classes of exact solutions to the nonlinear free boundary system (1). These exact solutions are constructed using various generating ordinary differential equations (ODEs) whose definitions are based on the initial conditions for (1). This approach has previously been applied to obtain solutions to certain nonlinear evolution equations arising in physics and chemistry [11, 12]. It is somewhat analogous to the method of undetermined coefficients for solving non-homogeneous ODEs with appropriate non-homogeneities. Throughout this work, the diffusivities are assumed to be affine-linear:

$$D_1(u,v) = \lambda_0^1 + \lambda_1^1 u + \lambda_2^1 v \qquad D_2(u,v) = \lambda_0^2 + \lambda_1^2 u + \lambda_2^2 v \tag{3}$$

where λ_i^k are constant. While this paper considers only the given two-component system, the solutions described here can certainly be adapted to similar single-component systems, or with appropriate assumptions, to other multi-component systems.

In the next section, a generalization of this approach is described for two-component systems such as (1) and (3) with time-dependent coefficients. In section 3 the approach is used to construct new solutions for the nonlinear evolution system (1) and (3). In section 4, we consider under what conditions and for what length of time the solutions derived in section 3 can satisfy the Stefan conditions of (2). A major result in this section is the finite-time blow-up of the interface velocity in some of the cases for certain parameter values.

2. A method for constructing exact solutions for a two-component nonlinear evolution system

To begin our study of system (1)–(3), let us first concentrate on the PDE (1) and consider the following linear homogeneous *generating system*:

$$\begin{aligned} &\alpha_0(t,x)U + \alpha_1(t,x)\frac{dU}{dx} + \cdots + \alpha_m(t,x)\frac{d^m U}{dx^m} = 0 \\ &\beta_0(t,x)V + \beta_1(t,x)\frac{dV}{dx} + \cdots + \beta_n(t,x)\frac{d^n V}{dx^n} = 0 \end{aligned} \tag{4}$$

where $\alpha_0(t,x),\dots,\alpha_m(t,x)$ and $\beta_0(t,x),\dots,\beta_n(t,x)$ are yet-to-be-specified (for the moment, arbitrary) continuous functions and the variable t is considered as a parameter. As we should see, the choice of α_i and β_i depends on the initial conditions for (1) and (3). It is well known that the general solution of system (4) has the form

$$\begin{aligned} U &= \varphi_0(t)g_0(t,x) + \cdots + \varphi_{m-1}(t)g_{m-1}(t,x) \\ V &= \psi_0(t)h_0(t,x) + \cdots + \psi_{n-1}(t)h_{n-1}(t,x) \end{aligned} \tag{5}$$

where $\varphi_0(t),\varphi_1(t),\dots,\varphi_{m-1}(t)$ and $\psi_0(t),\psi_1(t),\dots,\psi_{n-1}(t)$ are arbitrary functions and $g_0(t,x),\dots,g_{m-1}(t,x)$ and $h_0(t,x),\dots,h_{n-1}(t,x)$ is a fundamental set of solutions for (4). Note that in many important cases $g_0(t,x),\dots,g_{m-1}(t,x)$ and $h_0(t,x),\dots,h_{n-1}(t,x)$ can be expressed explicitly in terms of elementary functions. Moreover, in the case of time-dependent functions $\alpha_0(t),\dots,\alpha_m(t)$ and $\beta_0(t),\dots,\beta_n(t)$ a full list of fundamental sets of solutions can be written for all possible forms of these functions. So system (4) is an additional generating condition for obtaining many forms of the functions U and V.

Consider relations (5) as an ansatz for the PDEs (1) and (3). Note that this ansatz contains $m+n$ yet-to-be-determined functions (φ_i and ψ_j). There is no claim of completeness for the fundamental set generated by (4). However, given appropriate choices of α_i and β_i, this ansatz may reduce (1) to a quasilinear first-order system of ODEs for the unknown functions φ_i and ψ_j. Specifically, if one substitutes (5) into system (1) and (3) and regroups similar terms according to the powers of the functions $\varphi_i(t)$ and $\psi_j(t)$ and their derivatives, then it may be possible that the coefficients of these terms can be written as linear combinations of the linearly-independent elements of the fundamental set, and sufficient conditions for the reduction of this expression to a system of ODEs can be found. These sufficient conditions are given by (6)–(13), where $Q^k_{jj_1}(t)$, $R^k_{ii_1}(t)$, $S^{j_1,k}_{ij}(t)$ and $T^{j_2,k}_{jj_1}(t)$ on the right-hand side are independent of x and are defined by the expressions on the left-hand side. In other words, it is assumed that the expressions on the left-hand side of these conditions are some linear combinations (with respect to x) of the functions g_i and h_j with coefficients $Q^k_{jj_1}(t)$, $R^k_{ii_1}(t)$, $S^{j_1,k}_{ij}(t)$ and $T^{j_2,k}_{jj_1}(t)$. So the following conditions are found:

$$\lambda^1_0 g_{i,xx} + \mu g_{i,x} - g_{i,t} = g_{i_1} Q^1_{ii_1} \tag{6}$$
$$\lambda^2_0 h_{j,xx} + \mu h_{j,x} - h_{j,t} = h_{j_1} Q^2_{jj_1} \tag{7}$$
$$\lambda^1_1 g_i g_{i,xx} + \lambda^1_1 (g_{i,x})^2 = g_{i_1} R^1_{ii_1} \tag{8}$$
$$\lambda^2_2 h_j h_{j,xx} + \lambda^2_2 (h_{j,x})^2 = h_{j_1} R^2_{jj_1} \tag{9}$$
$$\lambda^1_1 (g_i g_{i_1,xx} + g_{i_1} g_{i,xx}) + 2\lambda^1_1 g_{i,x} g_{i_1,x} = g_{i_2} T^{i_2,1}_{ii_1} \qquad i < i_1 \tag{10}$$
$$\lambda^2_2 (h_j h_{j_1,xx} + h_{j_1} h_{j,xx}) + 2\lambda^2_2 h_{j,x} h_{j_1,x} = h_{j_2} T^{j_2,2}_{jj_1} \qquad j < j_1 \tag{11}$$
$$\lambda^1_2 h_j g_{i,xx} + \lambda^1_2 g_{i,x} h_{j,x} = g_{i_1} S^{i_1,1}_{ij} \tag{12}$$
$$\lambda^2_1 h_j g_{i,xx} + \lambda^2_1 g_{i,x} h_{j,x} = h_{j_1} S^{j_1,2}_{ij} \tag{13}$$

where on the right-hand sides of (6)–(13) a summation is assumed from 0 to $m-1$ over repeated indices i_1 and i_2, and from 0 to $n-1$ over j_1 and j_2.

Thus, assuming that it is possible to write the left-hand sides of (6)–(13) as linear combinations of the fundamental solutions, one obtains the following system of $(m+n)$ ODEs:

$$\begin{aligned}\frac{d\varphi_i}{dt} &= Q^1_{i_1 i}\varphi_{i_1} + R^1_{i_1 i}(\varphi_{i_1})^2 + T^{i,1}_{i_1 i_2}\varphi_{i_1}\varphi_{i_2} + S^{i,1}_{i_1 j_1}\varphi_{i_1}\psi_{j_1}\\ \frac{d\psi_j}{dt} &= Q^2_{j_1 j}\psi_{j_1} + R^2_{j_1 j}(\psi_{j_1})^2 + T^{j,2}_{j_1 j_2}\psi_{j_1}\psi_{j_2} + S^{j,2}_{i_1 j_1}\varphi_{i_1}\psi_{j_1}.\end{aligned} \tag{14}$$

Again, on the right-hand sides of (14), repeated indices are summed.

An important special case of the above approach may occur when $n = m$ and $\alpha_i = \beta_i$. In this case, φ_i and ψ_i can be multiples of each other, and the general solution of (4) is

$$\begin{aligned}U &= \varphi_0(t) g_0(t,x) + \cdots + \varphi_{m-1}(t) g_{m-1}(t,x)\\ V &= \psi_0(t) g_0(t,x) + \cdots + \psi_{m-1}(t) g_{m-1}(t,x)\end{aligned} \tag{15}$$

and if (15) is used as an ansatz for (1) and (3), then the equations

$$\begin{aligned}\frac{d\varphi_i}{dt} &= Q^1_{i_1 i}\varphi_{i_1} + R^1_{i_1 i}(\varphi_{i_1})^2 + T^{i,1}_{i_1 i_2}\varphi_{i_1}\varphi_{i_2}\\ \frac{d\psi_i}{dt} &= Q^2_{i_1 i}\psi_{i_1} + R^2_{i_1 i}(\psi_{i_1})^2 + T^{i,2}_{i_1 i_2}\psi_{i_1}\psi_{i_2}\end{aligned} \tag{16}$$

generate an exact solution in the form (15) for the nonlinear system (1) and (3) provided the functions g_i, $i = 0, \ldots, m-1$, satisfy the conditions

$$\lambda_0^1 g_{i,xx} + \mu g_{i,x} - g_{i,t} = g_{i_1} Q^1_{ii_1}(t) \tag{17}$$

$$\lambda_0^2 g_{i,xx} + \mu g_{i,x} - g_{i,t} = g_{i_1} Q^2_{ii_1}(t) \tag{18}$$

$$(\lambda_1^1 + \theta_i(t)\lambda_2^1)(g_i g_{i,xx} + (g_{i,x})^2) = g_{i_1} R^1_{ii_1}(t) \tag{19}$$

$$(\lambda_2^2 + (\lambda_1^2/\theta_i(t)))(g_i g_{i,xx} + (g_{i,x})^2) = g_{i_1} R^2_{ii_1}(t) \tag{20}$$

$$(\lambda_1^1 + \theta_i(t)\lambda_2^1) g_i g_{i_1,xx} + (\lambda_1^1 + \theta_{i_1}(t)\lambda_2^1) g_{i_1} g_{i,xx} + (2\lambda_1^1 + \theta_i(t)\lambda_2^1 + \theta_{i_1}(t)\lambda_2^1) g_{i,x} g_{i_1,x} = g_{i_2} T^{i_2,1}_{ii_1}(t) \qquad i < i_1 \tag{21}$$

$$\left(\lambda_2^2 + \frac{\lambda_1^2}{\theta_i(t)}\right) g_i g_{i_1,xx} + \left(\lambda_2^2 + \frac{\lambda_1^2}{\theta_{i_1}(t)}\right) g_{i_1} g_{i,xx} + \left(2\lambda_2^2 + \frac{\lambda_1^2}{\theta_i(t)} + \frac{\lambda_1^2}{\theta_{i_1}(t)}\right) g_{i,x} g_{i_1,x} = g_{i_2} T^{i_2,2}_{ii_1}(t) \qquad i < i_1 \tag{22}$$

where the functions $\theta_i(t) = \psi_i/\varphi_i$. The system of ODEs (16) is somewhat simpler since it does not contain the $S^{i,1}_{i_1 j_1}$ or $S^{i,2}_{i_1 j_1}$ terms. The functions $\theta_i(t)$ in relations (17)–(22) can be considered as some known functions for obtaining the functions Q, R, T with corresponding indices. Generally speaking, even in the case $\theta_i(t) = \theta_i \in \mathbb{R}$, one can obtain non-trivial solutions of the nonlinear system (1) and (3).

Proposition. Any solution of system (14) generates an exact solution in the form (5) for the nonlinear system (1) and (3), provided the functions g_i, $i = 0, \ldots, m-1$, and h_j, $j = 0, \ldots, n-1$, satisfy conditions (6)–(13). Similarly, any solution of system (16) generates an exact solution in the form (15) for the nonlinear system (1) and (3), provided the functions g_i, $i = 0, \ldots, m-1$, satisfy conditions (17)–(22).

Remark 1. This proposition also holds for the more general case $\lambda_i^k = \lambda_i^k(t)$.

Remark 2. The following nonlinear evolution system with arbitrary power nonlinearity $\alpha \in \mathbb{R}$

$$\begin{aligned} Y_t &= [(\lambda_0^1(t) + \lambda_1^1(t)Y + \lambda_2^1(t)Z^\alpha)Y_x]_x + \mu(t)Y_x \\ Z_t &= \lambda(t)(Z^\alpha Z_x)_x + \mu(t)Z_x \end{aligned} \tag{23}$$

where $Y = Y(t,x)$, $Z = Z(t,x)$ are unknown functions, is reduced by the substitution

$$u = Y \qquad v = Z^\alpha \qquad \alpha \neq 0 \tag{24}$$

to the system

$$\begin{aligned} u_t &= [\lambda_0^1(t) + \lambda_1^1(t)u + \lambda_2^1(t)v]u_{xx} + \lambda_1^1(t)u_x^2 + \lambda_2^1(t)u_x v_x + \mu(t)u_x \\ v_t &= \lambda(t)v v_{xx} + (\lambda(t)/\alpha)v_x^2 + \mu(t)v_x \end{aligned} \tag{25}$$

which has the form (1) and (3).

3. Exact solutions for the nonlinear evolution system

Let us now use the method described to construct several sets of exact solutions of system (1) and (3) assuming the coefficients λ_i^k are constant, namely

$$\begin{aligned} u_t &= [(\lambda_0^1 + \lambda_1^1 u + \lambda_2^1 v)u_x]_x + \mu(t)u_x \\ v_t &= [(\lambda_0^2 + \lambda_1^2 u + \lambda_2^2 v)v_x]_x + \mu(t)v_x. \end{aligned} \tag{26}$$

We also assume that $n = m = 3$ in the generating ODE system (4):

$$\alpha_1(t)\frac{\mathrm{d}U}{\mathrm{d}x} + \alpha_2(t)\frac{\mathrm{d}^2U}{\mathrm{d}x^2} + \frac{\mathrm{d}^3U}{\mathrm{d}x^3} = 0$$
$$\alpha_1(t)\frac{\mathrm{d}V}{\mathrm{d}x} + \alpha_2(t)\frac{\mathrm{d}^2V}{\mathrm{d}x^2} + \frac{\mathrm{d}^3V}{\mathrm{d}x^3} = 0. \tag{27}$$

Other choices of values for n and m are of course possible, but this choice contains enough terms to generate interesting solutions without forcing the algebra to be too tedious or requiring the assistance of a computer. The coefficient α_0 is assumed to be zero because we are seeking practically applicable solutions U, V that can be developed in series with the first terms $\varphi_0(t), \psi_0(t)$, respectively. Of course, the suggested method also works well for the case $\alpha_0 \neq 0$.

System (27) generates the following four ansätze:

(i) for $\alpha_1 = \alpha_2 = 0$,

$$U = \varphi_0(t) + \varphi_1(t)x + \varphi_2(t)x^2$$
$$V = \psi_0(t) + \psi_1(t)x + \psi_2(t)x^2 \tag{28}$$

(ii) for $\alpha_1 = 0$ and $\alpha_2 = -\gamma$,

$$U = \varphi_0(t) + \varphi_1(t)x + \varphi_2(t)\exp(\gamma(t)x)$$
$$V = \psi_0(t) + \psi_1(t)x + \psi_2(t)\exp(\gamma(t)x) \tag{29}$$

(iii) for $\gamma_{1,2}(t) = \frac{1}{2}(\pm(\alpha_2^2 - 4\alpha_1)^{1/2} - \alpha_2)$ and $\gamma_1 \neq \gamma_2$,

$$U = \varphi_0(t) + \varphi_1(t)\exp(\gamma_1(t)x) + \varphi_2(t)\exp(\gamma_2(t)x)$$
$$V = \psi_0(t) + \psi_1(t)\exp(\gamma_1(t)x) + \psi_2(t)\exp(\gamma_2(t)x) \tag{30}$$

(iv) finally if $\gamma_1 = \gamma_2 = \gamma \neq 0$ in this last case, then

$$U = \varphi_0(t) + \varphi_1(t)\exp(\gamma(t)x) + x\varphi_2(t)\exp(\gamma(t)x)$$
$$V = \psi_0(t) + \psi_1(t)\exp(\gamma(t)x) + x\psi_2(t)\exp(\gamma(t)x). \tag{31}$$

Substituting the functions $g_0 = h_0 = 1, g_1 = h_1 = x, g_2 = h_2 = x^2$ from ansatz (28) into relations (6)–(13), one obtains

$$Q_{10}^1 = Q_{10}^2 = \mu(t) \qquad Q_{20}^1 = 2\lambda_0^1$$
$$Q_{21}^1 = Q_{21}^2 = 2\mu(t) \qquad Q_{20}^2 = 2\lambda_0^2$$
$$R_{10}^1 = \lambda_1^1 \qquad R_{10}^2 = \lambda_2^2 \qquad R_{22}^1 = 6\lambda_1^1 \qquad R_{22}^2 = 6\lambda_2^2$$
$$T_{02}^{0,1} = 2\lambda_1^1 \qquad T_{02}^{0,2} = 2\lambda_2^2 \qquad T_{12}^{1,1} = 6\lambda_1^1 \qquad T_{12}^{1,2} = 6\lambda_2^2$$
$$S_{11}^{0,1} = \lambda_2^1 \qquad S_{20}^{0,1} = S_{12}^{1,1} = 2\lambda_2^1 \qquad S_{21}^{1,1} = 4\lambda_2^1 \qquad S_{22}^{2,1} = 6\lambda_2^1$$
$$S_{11}^{0,2} = \lambda_1^2 \qquad S_{02}^{0,2} = S_{21}^{1,2} = 2\lambda_1^2 \qquad S_{12}^{1,2} = 4\lambda_1^2 \qquad S_{22}^{2,2} = 6\lambda_1^2 \tag{32}$$

and

$$R_{ii_1}^k = Q_{ii_1}^k = T_{ii_1}^{j,k} = S_{ij}^{i_1,k} = 0 \tag{33}$$

for all combinations of the indices i, i_1, j, k not listed in (32). Then using (32) and (33), one finds that system (14) becomes

$$\frac{\mathrm{d}\varphi_0}{\mathrm{d}t} = \mu(t)\varphi_1 + 2\lambda_0^1\varphi_2 + \lambda_1^1(\varphi_1)^2 + 2\lambda_1^1\varphi_0\varphi_2 + \lambda_2^1\varphi_1\psi_1 + 2\lambda_2^1\varphi_2\psi_0$$
$$\frac{\mathrm{d}\varphi_1}{\mathrm{d}t} = 2\mu(t)\varphi_2 + 6\lambda_1^1\varphi_1\varphi_2 + 4\lambda_2^1\varphi_2\psi_1 + 2\lambda_2^1\varphi_1\psi_2$$

$$\frac{d\varphi_2}{dt} = 6\lambda_1^1(\varphi_2)^2 + 6\lambda_2^1\varphi_2\psi_2$$
$$\frac{d\psi_0}{dt} = \mu(t)\psi_1 + 2\lambda_0^2\psi_2 + \lambda_2^2(\psi_1)^2 + 2\lambda_2^2\psi_0\psi_2 + \lambda_1^2\varphi_1\psi_1 + 2\lambda_1^2\varphi_0\psi_2$$
$$\frac{d\psi_1}{dt} = 2\mu(t)\psi_2 + 6\lambda_2^2\psi_1\psi_2 + 4\lambda_1^2\varphi_1\psi_2 + 2\lambda_1^2\varphi_2\psi_1$$
$$\frac{d\psi_2}{dt} = 6\lambda_2^2(\psi_2)^2 + 6\lambda_1^2\varphi_2\psi_2. \tag{34}$$

System (34) is nonlinear, but fortunately it contains a simpler subsystem:

$$\frac{d\varphi_2}{dt} = 6\lambda_1^1(\varphi_2)^2 + 6\lambda_2^1\varphi_2\psi_2$$
$$\frac{d\psi_2}{dt} = 6\lambda_2^2(\psi_2)^2 + 6\lambda_1^2\varphi_2\psi_2. \tag{35}$$

There are many cases depending on the coefficients λ_i^k which lead to explicit solutions for (35) (two are given later). However, it is also possible to describe its solutions generically. Indeed, there are two cases: (1) both of the lines $\lambda_1^1\varphi_2 + \lambda_2^1\psi_2 = 0$ and $\lambda_1^2\varphi_2 + \lambda_2^2\psi_2 = 0$ lie in the same two quadrants of the phase plane, or (2) these lines lie in different quadrants. In either case, there is a quadrant where both $\lambda_1^1\varphi_2 < 0$ and $\lambda_2^2\psi_2 < 0$, and if the initial conditions lie in this quadrant, both φ_2 and ψ_2 converge to zero. If the initial conditions lie in any other quadrant, solutions grow unbounded in time. So for appropriate initial conditions, it is reasonable to set $\varphi_2 \equiv \psi_2 \equiv 0$ (this is also equivalent to assuming $n = m = 2$ in (4)). One can then write the solution to (34) as

$$\varphi_1 = u_1 \qquad \psi_1 = v_1$$
$$\varphi_0 = u_1 M(t) + (\lambda_1^1 u_1 + \lambda_2^1 v_1)u_1 t + u_0$$
$$\psi_0 = v_1 M(t) + (\lambda_1^2 u_1 + \lambda_2^2 v_1)v_1 t + v_0. \tag{36}$$

In (36) and hereinafter

$$M(t) := \int_0^t \mu(t)\, dt \qquad \Leftrightarrow \qquad \frac{dM}{dt} = \mu \qquad M(0) = 0 \tag{37}$$

and u_0, v_0, u_1, v_1 are arbitrary constants. So $M(t)$ is the interface position in the laboratory reference frame. Substituting functions (36) into ansatz (28), one obtains a four-parameter family of exact solutions of system (26)

$$u = u_0 + u_1[M(t) + (\lambda_1^1 u_1 + \lambda_2^1 v_1)t + x]$$
$$v = v_0 + v_1[M(t) + (\lambda_1^2 u_1 + \lambda_2^2 v_1)t + x]. \tag{38}$$

Let us assume that $\lambda_1^2 = 0$ and $\psi_2 = 0$. In this case it is easy to construct the general solution of system (34) since subsystem (35) is reduced to the one simple ODE. So the following functions are found:

$$\varphi_0 = \varphi(t) \qquad \psi_0 = -v_1 M(t) + v_1^2\lambda_2^2 t + v_0$$
$$\varphi_1 = \frac{1}{3\lambda_1^1}\left[\exp\varphi_2 \int \mu(t)\exp(-\varphi_2)\, d\varphi_2 + v_0\exp\varphi_2 + 2v_1\lambda_2^1\right] \qquad \text{if } \lambda_1^1 \neq 0$$
$$\varphi_1 = -2u_2 M(t) + 4u_2 v_1\lambda_2^1 t + u_0 \qquad \text{if } \lambda_1^1 = 0$$
$$\psi_1 = -v_1 \qquad \varphi_2 = -\frac{u_2}{1 + 6u_2\lambda_1^1 t} \tag{39}$$

where $\varphi(t)$ is the general solution of the linear first-order ODE

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} - 2\lambda_1^1\varphi_2\varphi = (\mu(t) - \lambda_2^1 v_1)\varphi_1 + \lambda_1^1(\varphi_1)^2 + 2[\lambda_0^1 + \lambda_2^1(-v_1 M(t) + v_1^2\lambda_2^2 t + v_0)]\varphi_2. \quad (40)$$

Substituting functions (39) into ansatz (28), one obtains a five-parameter family of exact solutions of system (26) at $\lambda_1^2 = 0$

$$u = \varphi(t) + \varphi_1(t)x - \frac{u_2}{1 + 6u_2\lambda_1^1 t}x^2$$
$$v = v_0 + v_1^2\lambda_2^2 t - v_1(M(t) + x). \quad (41)$$

Analogous to how ansatz (28) was used, one can substitute the functions g_i and h_i from ansatz (29) into relations (6)–(13), obtaining the corresponding values of the functions $R_{ii_1}^k$, $Q_{ii_1}^k$, $T_{ii_1}^{j,k}$ and $S_{ij}^{i_1,k}$. However, then one finds the constraint $\varphi_2 = \psi_2 = 0$, leading to the same solutions as in (38). For this reason, we use relations (17)–(22) to construct solutions for which $\varphi_2\psi_2 \neq 0$. In fact, assuming that γ is constant, and substituting the functions $g_0 = h_0 = 1$, $g_1 = h_1 = x$, $g_2 = h_2 = \exp(\gamma x)$ from ansatz (29) into relations (17)–(22), one obtains values for the functions $R_{ii_1}^k$, $Q_{ii_1}^k$ and $T_{ii_1}^{j,k}$ which reduce system (16) to

$$\frac{\mathrm{d}\varphi_0}{\mathrm{d}t} = \mu(t)\varphi_1 \qquad \frac{\mathrm{d}\varphi_1}{\mathrm{d}t} = 0$$
$$\frac{\mathrm{d}\varphi_2}{\mathrm{d}t} = \gamma^2(\lambda_0^1 + \lambda_1^1\varphi_0 + \lambda_2^1\psi_0)\varphi_2 + \gamma\mu(t)\varphi_2$$
$$\frac{\mathrm{d}\psi_0}{\mathrm{d}t} = \mu(t)\psi_1 \qquad \frac{\mathrm{d}\psi_1}{\mathrm{d}t} = 0$$
$$\frac{\mathrm{d}\psi_2}{\mathrm{d}t} = \gamma^2(\lambda_0^2 + \lambda_1^2\varphi_0 + \lambda_2^2\psi_0)\psi_2 + \gamma\mu(t)\psi_2 \quad (42)$$

assuming that for some constant $\alpha \in \mathbb{R}$, the following additional constraints are also satisfied:

$$\lambda_1^2 = \alpha\lambda_1^1 \qquad \lambda_2^2 = \alpha\lambda_2^1 \qquad \psi_l = -(\lambda_1^1/\lambda_2^1)\varphi_l \qquad l = 1, 2 \qquad (\theta_l(t) \equiv -\lambda_1^1/\lambda_2^1)$$
$$\lambda_0^2 = \lambda_0^1 + (1 - \alpha)(\lambda_1^1\varphi_0 + \lambda_2^1\psi_0). \quad (43)$$

The system of ODEs (42) is integrable, and its general solution can be written explicitly. Specifically, one obtains the following three-parameter family of exact solutions to system (26),

$$u = u_0 + u_1[M(t) + x] + u_2\exp(N(t) + \gamma x)$$
$$v = v_0 - (\lambda_1^1/\lambda_2^1)u_1[M(t) + x] - (\lambda_1^1/\lambda_2^1)u_2\exp(N(t) + \gamma x) \quad (44)$$

where the diffusivities are of the form

$$D_1(u, v) = \lambda_0^1 + \lambda_1^1 u + \lambda_2^1 v$$
$$D_2(u, v) = \lambda_0^1 + (1 - \alpha)(\lambda_1^1 u_0 + \lambda_2^1 v_0) + \alpha(\lambda_1^1 u + \lambda_2^1 v) \quad (45)$$

u_1, u_2 and γ are arbitrary constants, and $N(t) := \gamma^2 D_0 t + \gamma M(t)$. Note that for these solutions the diffusivities are constant and equal. Hence one can define

$$D_0 := \lambda_0^1 + \lambda_1^1 u_0 + \lambda_2^1 v_0 \equiv D_1(u(x, t), v(x, t)) = D_2(u(x, t), v(x, t)). \quad (46)$$

In the case where $\gamma(t)$ is not constant, ansatz (29) also gives an exact solution of system (26) with the diffusivities given by

$$D_1 = D_2 = D(u, v) \equiv \lambda_0^1 + \lambda_1^1 u + \lambda_2^1 v. \quad (47)$$

Indeed, in this case $g_{2,t} = \gamma^{-2}(\mathrm{d}\gamma/\mathrm{d}t)g_1 g_{2,xx}$, so that one can transfer this term from relations (17) and (18), $i = 2$, into (21) and (22), $i = 1, i_1 = 2$, respectively. The family of exact solutions is then

$$u = \varphi_0(t) + u_1 x + \varphi_2(t)\exp(\gamma(t)x)$$
$$v = \psi_0(t) + v_1 x - (\lambda_1^1/\lambda_2^1)\varphi_2(t)\exp(\gamma(t)x). \tag{48}$$

In (48) the function $\gamma = (u_2 - (\lambda_1^1 u_1 + \lambda_2^1 v_1)t)^{-1}$, and the functions φ_0, ψ_0 and φ_2 are solutions of

$$\frac{\mathrm{d}\varphi_0}{\mathrm{d}t} = u_1(\mu(t) + \lambda_1^1 u_1 + \lambda_2^1 v_1)$$
$$\frac{\mathrm{d}\psi_0}{\mathrm{d}t} = v_1(\mu(t) + \lambda_1^1 u_1 + \lambda_2^1 v_1)$$
$$\frac{\mathrm{d}\varphi_2}{\mathrm{d}t} = \gamma(t)[\mu(t) + \lambda_1^1 u_1 + \lambda_2^1 v_1 + \gamma(t)D(\varphi_0, \psi_0)]\varphi_2. \tag{49}$$

Note that this system is clearly integrable.

Remark 3. The family of exact solutions (48) has an essential difference from the ones obtained above or in [11, 12]: it contains a non-constant $\gamma(t)$. This family cannot be obtained using the approach recently proposed in [13, 14] (note that the basic ideas of the approach used in [13, 14] are present in [15]).

Next, substituting the functions $g_0 = h_0 = 1$, $g_1 = h_1 = \exp(\gamma_1 x)$, $g_2 = h_2 = \exp(\gamma_2 x)$ from ansatz (30) into relations (17)–(22) (γ_1 and γ_2 being constant), one obtains values of the functions $R_{ii_1}^k$, $Q_{ii_1}^k$, $T_{ii_1}^{j,k}$ and $S_{ij}^{i_1,k}$ for which system (16) has the form

$$\frac{\mathrm{d}\varphi_0}{\mathrm{d}t} = 0 \qquad \frac{\mathrm{d}\psi_0}{\mathrm{d}t} = 0$$
$$\frac{\mathrm{d}\varphi_i}{\mathrm{d}t} = \gamma_i\mu(t)\varphi_i + \gamma_i^2[\lambda_0^1\varphi_i + \lambda_1^1\varphi_0\varphi_i + \lambda_2^1\psi_0\varphi_i] \qquad i = 1, 2 \tag{50}$$

with the additional constraints (43). System (50) is easily integrated, and substituting its general solution into ansatz (30), one obtains the following four-parameter family of exact solutions of system (26)

$$u = u_0 + u_1\exp[\gamma_1^2 D_0 t + \gamma_1(M(t) + x)] + u_2\exp[\gamma_2^2 D_0 t + \gamma_2(M(t) + x)]$$
$$v = v_0 - (\lambda_1^1/\lambda_2^1)u_1\exp[\gamma_1^2 D_0 t + \gamma_1(M(t) + x)]$$
$$\qquad - (\lambda_1^1/\lambda_2^1)u_2\exp[\gamma_2^2 D_0 t + \gamma_2(M(t) + x)] \tag{51}$$

where u_1, u_2, γ_1 and γ_2 are arbitrary constants.

Note that if $\gamma_1 = -\gamma_2 = \mathrm{i}\gamma$, $\gamma \in \mathbb{R}$, $\mathrm{i}^2 = -1$ and $u_1 = u_2 = c/2$, any complex solution of the form (51) generates a real oscillatory solution, namely

$$u = u_0 + c\exp[-\gamma^2 D_0 t]\cos[\gamma(M(t) + x)]$$
$$v = v_0 - (\lambda_1^1/\lambda_2^1)c\exp[-\gamma^2 D_0 t]\cos[\gamma(M(t) + x)]. \tag{52}$$

On the other hand, the family of exact solutions (51) admits the following non-trivial generalization

$$u = u_0 + u_k\exp[\gamma_k^2 D_0 t + \gamma_k(M(t) + x)]$$
$$v = v_0 - (\lambda_1^1/\lambda_2^1)u_k\exp[\gamma_k^2 D_0 t + \gamma_k(M(t) + x)] \tag{53}$$

where a summation from 1 to n is assumed over the repeated indices k, and u_k and γ_k are arbitrary constants.

Finally, using ansatz (31) and relations (17)–(22), one again obtains $R^k_{ii_1}$, $Q^k_{ii_1}$, $T^{j,k}_{ii_1}$ and $S^{i_1,k}_{ij}$, for which system (16) has the form

$$\frac{d\varphi_0}{dt} = 0 \qquad \frac{d\psi_0}{dt} = 0$$
$$\frac{d\varphi_1}{dt} = \mu(t)(\gamma\varphi_1 + \varphi_2) + \gamma[\lambda^1_0 + \lambda^1_1\varphi_0 + \lambda^1_2\psi_0](\gamma\varphi_1 + 2\varphi_2)$$
$$\frac{d\varphi_2}{dt} = \gamma\mu(t)\varphi_2 + \gamma^2[\lambda^1_0 + \lambda^1_1\varphi_0 + \lambda^1_2\psi_0]\varphi_2 \tag{54}$$

with the additional constraints (43). One can again obtain a four-parameter family of exact solutions of system (26):

$$u = u_0 + [u_1 + 2u_2\gamma D_0 t + u_2(M(t) + x)]\exp[\gamma^2 D_0 t + \gamma(M(t) + x)]$$
$$v = v_0 - (\lambda^1_1/\lambda^1_2)[u_1 + 2u_2\gamma D_0 t + u_2(M(t) + x)]\exp[\gamma^2 D_0 t + \gamma(M(t) + x)]. \tag{55}$$

As in the case of solution (51), one can generalize this solution to a family that contains arbitrary constants u_k and γ_k, $k = 1, \ldots, n$.

Remark 4. While the diffusivities are affine-linear in u and v, in some cases considered above the component parameters for the exact solutions (see formulae (44), (51)–(53), (55)) are such that along these exact solutions, the diffusivities are constant in space and time. On the other hand, the solutions (38), (41) and (48) do not share this property; for these solutions the diffusivities vary in space and time.

4. Precipitant-assisted protein crystal growth

This final section considers which of the exact solutions given above also satisfy the Stefan conditions of (2) at the interface and are therefore meaningful in connection with the precipitant-assisted protein crystal growth model. The equilibrium condition $f(u, v) = 0$, however, is not considered here in detail. Regarding the equilibrium condition $f(u, v) = 0$ at the interface $x = 0$, we can find for each solution discussed below that $(u(0), v(0))$ is a root of the linear f with time-independent coefficients. That is, our solutions are special in that only for certain equilibrium functions do they occur. In each case we can see that f satisfies whatever conditions are necessary to support the special solutions. So in each case, the solutions will satisfy (1) and (3), the Stefan and equilibrium conditions of (2), along with the stated initial conditions.

Let us consider the four-parameter family (38). This family requires that the initial protein and precipitant profiles be linearly ramped:

$$u(x, 0) = u_0 + u_1 x \qquad v(x, 0) = v_0 + v_1 x. \tag{56}$$

The most significant result in the case of these linearly-ramped initial profiles is the finite-time blow-up of the interface velocity for certain values of the system parameters (u_0, v_0, u_1, v_1, u_s, v_s) and the diffusivities. Substituting (38) into the Stefan conditions of (2) and assuming that $u_1 \neq 0$, $v_1 \neq 0$, one finds that the interface velocity μ must satisfy

$$\mu\left[\frac{u_s - u_0}{u_1} - M(t) - (\lambda^1_1 u_1 + \lambda^1_2 v_1)t\right] = \lambda^1_0 + \lambda^1_1 u_0 + \lambda^1_2 v_0$$
$$+(\lambda^1_1 u_1 + \lambda^1_2 v_1)M(t) + [\lambda^1_1(\lambda^1_1 u_1 + \lambda^1_2 v_1) + \lambda^1_2(\lambda^2_1 u_1 + \lambda^2_2 v_1)]t$$
$$\mu\left[\frac{v_s - v_0}{v_1} - M(t) - (\lambda^2_1 u_1 + \lambda^2_2 v_1)t\right] = \lambda^2_0 + \lambda^2_1 u_0 + \lambda^2_2 v_0$$
$$+(\lambda^2_1 u_1 + \lambda^2_2 v_1)M(t) + [\lambda^2_1(\lambda^1_1 u_1 + \lambda^1_2 v_1) + \lambda^2_2(\lambda^2_1 u_1 + \lambda^2_2 v_1)]t \tag{57}$$

where again $M(t) := \int_0^t \mu\,\mathrm{d}t$. Although there may be other interesting solutions, probably the easiest non-trivial one occurs when $u_1 = -v_1$, $u_s - u_0 = v_0 - v_s$, and $\lambda_k^1 = \lambda_k^2$, $k = 1, 2, 3$. Hence one can again define $D_0 := D_1(u_0, v_0) = D_2(u_0, v_0)$. The first of these conditions ($u_1 = -v_1$) is reasonable since one would expect that increasing (decreasing) the protein concentration would correspond to decreasing (increasing) the precipitant concentration. The other compatibility conditions are what is then required to make the two equations in (57) redundant. The requirement that the two diffusivities be equal for (u_0, v_0) is perhaps least realistic. These assumptions lead to the following particular case of the solution (38):

$$\begin{aligned} u &= u_0 + u_1[M(t) + (\lambda_1^1 - \lambda_2^1)u_1 t + x] \\ v &= v_0 - u_1[M(t) + (\lambda_1^1 - \lambda_2^1)u_1 t + x]. \end{aligned} \tag{58}$$

Both equations from (57) are then in the form of a nonlinear ODE:

$$\frac{\mathrm{d}M}{\mathrm{d}t}[A - (M + Bt)] = D_0 + BM + Ct \tag{59}$$

where A, B, and C are the appropriate constant groupings. Defining $q := A - (M + Bt)$ and noting that $q(0) = A$, one can integrate (51) to find that

$$q(t) = [A^2 - 2(AB + D_0)t - (C - B^2)t^2]^{1/2} \tag{60}$$

and hence that

$$\mu(t) = -B + \frac{(AB + D_0) + (C - B^2)t}{(A^2 - 2(AB + D_0)t - (C - B^2)t^2)^{1/2}}. \tag{61}$$

Therefore, depending on the values of A, B, C and D_0, the velocity may become infinite in finite time. Based on [1], one would expect that $A = (u_s - u_0)/u_1 > 0$ and $D_0 > 0$, but the signs of B and C may be positive or negative. Since $u_1 = -v_1$, the grouping $C - B^2 = (\lambda_1^1 - \lambda_2^1)(\lambda_1^1(1 - u_1) + \lambda_2^1(1 + u_1))u_1$, and this grouping is positive in many cases of interest (e.g., example 2 in [2]). When $C - B^2 > 0$, the maximum time for which this solution is valid is

$$\begin{aligned} t_{\max} = {}& [-\lambda_1^1 u_s - \lambda_2^1 v_s - \lambda_0^1 + ((\lambda_1^1 u_s + \lambda_2^1 v_s + \lambda_0^1)^2 + (u_s - u_0)^2(\lambda_1^1 - \lambda_2^1)(\lambda_1^1(1 - u_1) \\ & + \lambda_2^1(1 + u_1))/u_1)^{1/2}][u_1(\lambda_1^1 - \lambda_2^1)(\lambda_1^1(1 - u_1) + \lambda_2^1(1 + u_1))]^{-1}. \end{aligned} \tag{62}$$

In other words, in the case $C - B^2 > 0$ a blow-up regime is obtained, i.e. the velocity tends to infinity in finite time [10]. Of course, for other values of A, B and C the interface velocity $\mu(t)$ may be finite for all time.

It turns out that the triplet (u, v, μ) of the form (58) and (61) does not lose stability for sufficiently small variations of initial conditions (56). Indeed, consider the bounded domain $G = \langle (t, x) \in [0, T] \times [0, X] \rangle$, where $T < t_{\max}$ and X is determined from the natural constraints $u \geqslant 0$, $v \geqslant 0$ (see (58)).

Let us assume a small variation in (56) of the form

$$u'(x, 0) = u_0 + \epsilon_{10} + (u_1 + \epsilon_{11})x \qquad v'(x, 0) = v_0 + \epsilon_{20} - (u_1 + \epsilon_{21})x \tag{63}$$

where ϵ_{10}, ϵ_{11}, ϵ_{20}, ϵ_{21} are some small parameters, i.e. $|\epsilon_{ij}| \ll 1$. Renaming $u_0 + \epsilon_{10} = u_0^*$, $u_1 + \epsilon_{11} = u_1^*$, $v_0 + \epsilon_{20} = v_0^* + \epsilon_0$, $\epsilon_{21} - \epsilon_{11} = \epsilon_1$, $\epsilon_{10} + \epsilon_{20} = \epsilon_0$, one obtains from (63) ($*$ is omitted below)

$$u'(x, 0) = u_0 + u_1 x \qquad v'(x, 0) = v_0 + \epsilon_0 - (u_1 + \epsilon_1)x. \tag{64}$$

Having (38), one can check that the solution

$$\begin{aligned} u' &= u_0 + u_1[M'(t) + (\lambda_1^1 - \lambda_2^1)u_1 t - \lambda_2^1\epsilon_1 t + x] \\ v' &= v_0 + \epsilon_0 - (u_1 + \epsilon_1)[M'(t) + (\lambda_1^1 - \lambda_2^1)u_1 t - \lambda_2^1\epsilon_1 t + x] \end{aligned} \tag{65}$$

satisfies the initial condition (64). So, for sufficiently small non-zero ϵ_0 and ϵ_1 the following estimation is true:

$$\begin{aligned}|u' - u| &= |u_1\lambda_2^1\epsilon_1 t| \leqslant |u_1\lambda_2^1\epsilon_1 T| < \epsilon \\ |v' - v| &= |\epsilon_0 + u_1\lambda_2^1\epsilon_1 t - \epsilon_1[M'(t) + (\lambda_1^1 - \lambda_2^1)u_1 t - \lambda_2^1\epsilon_1 t + x]| \\ &\leqslant |\epsilon_0| + |u_1\lambda_2^1\epsilon_1 T| + |\epsilon_1|[|M'(T)| + |(\lambda_1^1 - \lambda_2^1)u_1 T| + |\lambda_2^1\epsilon_1 T| + |X|] < \epsilon\end{aligned} \tag{66}$$

where $\epsilon > 0$ is a given small parameter.

Using (65) and the first Stefan conditions in (2), the following ODE for M' is obtained:

$$\frac{\mathrm{d}M'}{\mathrm{d}t}[A' - (M' + B't)] = D_0' + B'M' + C't \tag{67}$$

where $A' = A$, $B' = B - \lambda_2^1\epsilon_1$, $C' = C - (\lambda_1^1 + \lambda_2^1)\lambda_2^1\epsilon_1$ and $D_0' = D_0 + \lambda_2^1\epsilon_0$. Therefore, the constraint

$$C' - B'^2 = C - B^2 + \epsilon_1\lambda_2^1[2u_1(\lambda_1^1 - \lambda_2^1) - (\lambda_1^1 + \lambda_2^1)] - (\lambda_2^1\epsilon_1)^2 > 0 \tag{68}$$

is again true for sufficiently small ϵ_1. Taking into account (67)–(68) we find

$$\mu'(t) = -B' + \frac{(AB' + D_0') + (C' - B'^2)t}{(A^2 - 2(AB' + D_0')t - (C' - B'^2)t^2)^{1/2}} \tag{69}$$

for which the estimation

$$|\mu' - \mu| < \epsilon \tag{70}$$

is true for sufficiently small non-zero ϵ_0 and ϵ_1.

Now the second Stefan condition in (2) can be reduced to the form

$$\mu'\left[\frac{u_s - u_0}{u_1} + \frac{v_s - v_0 - \epsilon_0}{u_1 + \epsilon_1}\right] = 0 \tag{71}$$

and then using $u_s - u_0 = v_0 - v_s$ one finds

$$\mu'\left[\frac{\epsilon_1(u_s - u_0) - \epsilon_0 u_1}{u_1(u_1 + \epsilon_1)}\right] = 0. \tag{72}$$

Again it is clear that for sufficiently small non-zero ϵ_0 and ϵ_1 it is possible to obtain the estimation

$$|\mu'|\left|\left[\frac{\epsilon_1(u_s - u_0) - \epsilon_0 u_1}{u_1(u_1 + \epsilon_1)}\right]\right| < \epsilon \tag{73}$$

so that the second Stefan condition is valid up to the given ϵ.

Finally, the equilibrium condition for the triplet (u', v', μ') has the form

$$f(u', v') := u'\left(1 + \frac{\epsilon_1}{u_1}\right) + v' - u_0\left(1 + \frac{\epsilon_1}{u_1}\right) - v_0 - \epsilon_0 = 0. \tag{74}$$

So the estimation

$$|f(u', v') - f(u, v)| = \left|\epsilon_1\frac{u' - u_0}{u_1} - \epsilon_0\right| < \epsilon \tag{75}$$

is proved again.

Thus the estimations (66), (70) and (75) give the following statement of stability of solution (58) and (61).

Statement. For any small $\epsilon > 0$ and positive $C - B^2$, sufficiently small non-zero ϵ_0 and ϵ_1 exist such that the exact solution (58) and (61) of the Stefan problem (1)–(3) and (56) is stable in the bounded domain G with respect to the small variations of the initial conditions (64).

Remark 5. Constraint (68) is very important in the case $0 < C - B^2 \ll 1$ since for the unsufficiently small non-zero ϵ_1 one can obtain $C' - B'^2 < 0$ and then estimation (70) will be wrong. So, in the case $C - B^2 = 0$ (e.g., $\lambda_1^1 = \lambda_2^1$) solutions (58) and (61) are unstable with respect to the small variations of the initial conditions (64).

For the found family of solutions (41) one can consider the following initial profiles:

$$u = u_0 + u_1 x - u_2 x^2 \qquad v = v_0 - v_1 x \tag{76}$$

which are a generalization of (56). It turns out that it is also possible to construct the solutions of the problem (1)–(3) and (76). The form of these solutions again essentially depends on the diffusivities D_1 and D_2. Although there are many other solutions, probably the most interesting non-trivial one occurs when $D_1 = \lambda_0^1 + \lambda_2^1 v, \quad D_2 = 2D_1, \lambda_0^1 = -\lambda_2^1 v_s$. For such diffusivities the *time-independent solution* is obtained:

$$\begin{aligned} u &= u_s - u_2 \left(x - \frac{v_0 - v_s}{v_1} \right)^2 \\ v &= v_0 - v_1 x \qquad \mu = 2\lambda_2^1 v_1. \end{aligned} \tag{77}$$

In other words, the initial profiles of concentrations move with constant velocity. Since the diffusivities must be non-negative, the maximum space for which solution (77) is valid are found:

$$x_{\max} = \frac{v_0 - v_s}{v_1} \qquad \text{if } u_s v_1^2 - u_2 (v_0 - v_s)^2 > 0 \qquad \lambda_2^1 v_1 > 0.$$

Note also that along the solution (77) the diffusivities D_1 and D_2 are not constant.

Now let us consider the three-parameter family (44). First suppose that the initial protein and precipitant profiles have exponential terms, but no linear terms (i.e. $u_1 = 0$):

$$u(x,0) = u_0 + u_2 \exp(\gamma x) \qquad v(x,0) = v_0 - (\lambda_1^1/\lambda_2^1) u_2 \exp(\gamma x). \tag{78}$$

In this case, the velocity may again blow up in finite time. Since only a decaying exponential would be physically reasonable, assume that $\gamma < 0$. Defining again $M(t) := \int_0^t \mu \, \mathrm{d}t$, substituting the protein and precipitant profiles (44) into the Stefan conditions of (2), and assuming again appropriate compatibility conditions (e.g., $u_2 = -v_2$, $u_0 - u_s = v_s - v_0$, etc), so that the Stefan conditions are redundant, one finds that the interface position M must satisfy the ODE

$$\gamma \frac{\mathrm{d}M}{\mathrm{d}t} [A - \exp(\beta t + \gamma M)] = \beta \exp(\beta t + \gamma M) \tag{79}$$

where now $A = (u_s - u_0)/u_2$ and $\beta := \gamma^2 D_0 > 0$. Integrating (79), one finds that M must satisfy

$$A\gamma M = \exp(\beta t + \gamma M) - 1. \tag{80}$$

Equation (80) implicitly defines the interface position $M(t)$. Consider a plot of both the right-hand side and left-hand side as a function of M with t as a parameter. For $A \leqslant 0$, the left-hand side is linear with non-negative slope, while the right-hand side is a decreasing exponential. Hence there is a unique positive solution $M(t)$ for all t. On the other hand, for $0 < A < 1$, the slope of the left-hand side is negative, and for a finite range of time, the left- and right-hand sides cross twice. At the largest such time $t_{\max}$, the left- and right-hand sides have a common slope at their single point of intersection; hence

$$t_{\max} = \frac{1}{\gamma^2 D_0}\left(\ln\left(\frac{u_s - u_0}{u_2}\right) + \frac{u_2 - u_s + u_0}{u_s - u_0}\right). \tag{81}$$

Looking back at (79), one sees that indeed $\mu(t)$ becomes unbounded at $t = t_{\max}$. This means that in the case $0 < A < 1$, a blow-up regime again is obtained. For $A \geqslant 1$, there is no solution for $t > 0$.

Next consider fully the three-parameter family (44) where the initial protein and precipitant profiles have both exponential and linear terms:

$$\begin{aligned} u(x,0) &= u_0 + u_1 x + u_2 \exp(\gamma x) \\ v(x,0) &= v_0 - (\lambda_1^1/\lambda_2^1) u_1 x - (\lambda_1^1/\lambda_2^1) u_2 \exp(\gamma x). \end{aligned} \tag{82}$$

In this case the interface behaviour may be rather more complicated. Again assume that $\gamma < 0$; under similar compatibility assumptions as before, the interface position M now must satisfy the ODE

$$\gamma \frac{\mathrm{d}M}{\mathrm{d}t}[A - (M + E\exp(\beta t + \gamma M))] = \beta(1/\gamma + E\exp(\beta t + \gamma M)) \tag{83}$$

where $E = u_2/u_1$ and again $\beta = \gamma^2 D_1^0 > 0$ and $A = (u_s - u_0)/u_1$. Unfortunately, this equation cannot be solved explicitly; it can, however, be rescaled to eliminate two of the parameters. Recalling that $-\gamma > 0$, replace $-\gamma M$ by M, βt by t, $-\gamma A$ by A and $-\gamma E$ by E. Then (83) becomes

$$\frac{\mathrm{d}M}{\mathrm{d}t}[A - (M + E\exp(t - M))] = 1 - E\exp(t - M). \tag{84}$$

The behaviour of the interface is then determined through (84) by the relative values or A and E. Numerical computations and a study of (84) indicate that there are three generic cases.

- For E sufficiently large and A sufficiently small or negative, $\mu > 0$ for all t. Hence the solid phase grows for all time.
- For E sufficiently small and A sufficiently large, the interface velocity will again blow up in finite time.
- For E sufficiently small and A sufficiently small or negative, the interface velocity is initially negative, but then changes sign as the numerator passes through a zero. After this, the solid phase grows for all time.

It would be interesting in the future to consider the case of initial conditions (82) more thoroughly.

Now let us consider the four-parameter family (55); this family requires the following initial conditions:

$$u(x,0) = u_0 + (u_1 + u_2 x)\exp(\gamma x) \qquad v(x,0) = v_0 - (\lambda_1^1/\lambda_2^1)(u_1 + u_2 x)\exp(\gamma x) \tag{85}$$

where again $\gamma < 0$. Conditions (85) are a generalization of (78). On substituting this four-parameter family (55) into the Stefan conditions, one finds that in this case $M(t) := \int_0^t \mu \,\mathrm{d}t$ must satisfy

$$\begin{aligned} (u_s - u_0)\frac{\mathrm{d}M}{\mathrm{d}t} = {} & \left[\left(\gamma D_0 + \frac{\mathrm{d}M}{\mathrm{d}t}\right)(u_1 + 2u_2\gamma D_0 t + u_2 M(t)) + D_0 u_2\right] \\ & \times \exp(\gamma^2 D_0 t + \gamma M(t)). \end{aligned} \tag{86}$$

This equation is not explicitly integrable for $u_s \neq u_0$. One could analyse this equation as we did in the previous case, but it is perhaps more interesting to note that if $u_2 = -\gamma(u_s - u_0)$ (86) has an explicit exact solution $M(t) = -\gamma D_0 t$ satisfying the initial condition $M(0) = 0$. The corresponding velocity is of course constant $\mu = -\gamma D_0$ and $D_0 > 0$.

In the case $u_s = u_0$, equation (86) with the initial condition $M(0) = 0$ gives a solution in the implicit form

$$\gamma^2 D_0 t + \gamma M(t) = -\ln\left|1 + \frac{u_2}{\gamma u_1 - u_2}(2\gamma^2 D_0 t + \gamma M(t))\right| \qquad \gamma \neq u_2/u_1. \tag{87}$$

Note that if $u_2/(\gamma u_1 - u_2) < 0$, there is again a maximum time for which this solution is valid.

Finally, we consider a simplification of the nonlinear system (1) with diffusivities (45) that is implied by remark 4. If we consider the initial conditions

$$u(x,0) = u_0 + w_0(x) \qquad v(x,0) = v_0 - (\lambda_1^1/\lambda_2^1) w_0(x) \tag{88}$$

where $w_0(x)$ is an arbitrary smooth function, then this system can be reduced to a single linear equation. Indeed, putting

$$u(x,t) = u_0 + w(x,t) \qquad v(x,t) = v_0 - (\lambda_1^1/\lambda_2^1) w(x,t) \tag{89}$$

one reduces problem (1), (45) and (48) to the linear Cauchy problem

$$w_t = D_0 u_{xx} + \mu w_x \qquad w(x,0) = w_0(x) \tag{90}$$

where $D_0 := \lambda_0^1 + \lambda_1^1 u_0 + \lambda_2^1 v_0 \equiv D_1(u(x,t), v(x,t)) = D_2(u(x,t), v(x,t))$. It is well known that the convection heat equation in (90) is reduced to the classical heat equation by the substitution $x' = x + M(t)$. So it is easy to find the fundamental solution of this convection heat equation

$$G(x,t) = \frac{1}{\sqrt{4\pi D_0 t}} \exp\left[-\frac{(x + M(t))^2}{4 D_0 t}\right]. \tag{91}$$

Thus the solution of the Cauchy problem (90) is written as the Poisson-type integral, namely

$$w(x,t) = \int_{-\infty}^{\infty} G(x-y,t) w_0(y)\,\mathrm{d}y. \tag{92}$$

Substituting the right-hand side of (92) into (89) one finds the solution of the initial problem (1), (45) and (88).

Finally, consider the interface conditions (2). It is easily seen that the solution (89) and (92) satisfy the following equilibrium condition

$$f(u,v) := \lambda_1^1 u + \lambda_2^1 v - \lambda_1^1 u_0 - \lambda_2^1 v_0 = 0. \tag{93}$$

After substituting (89) into the Stefan conditions of (2), they can be reduced to the form

$$\mu(u_s - u_0 - w) = D_0 w_x \qquad \lambda_1^1(u_s - u_0) = \lambda_2^1(v_0 - v_s) \tag{94}$$

where the second condition is the fixed constraint for the given coefficients. Taking into account formulae (91) and (92), we can represent the first condition of (94) as the nonlinear integro-differential equation

$$\frac{\mathrm{d}M}{\mathrm{d}t}(u_s - u_0 - I_0(t,M)) = \frac{1}{2t}[I_1(t,M) - M I_0(t,M)] \tag{95}$$

where $M(0) = 0$ and

$$I_0(t, M) = \frac{1}{\sqrt{4\pi D_0 t}} \int_{-\infty}^{\infty} \exp\left[-\frac{(y - M(t))^2}{4D_0 t}\right] w_0(y)\,\mathrm{d}y$$
$$I_1(t, M) = \frac{1}{\sqrt{4\pi D_0 t}} \int_{-\infty}^{\infty} \exp\left[-\frac{(y - M(t))^2}{4D_0 t}\right] w_0(y)\,y\,\mathrm{d}y. \tag{96}$$

It is clear that it is impossible to construct an exact solution of this nonlinear integro-differential equation for an arbitrary smooth function w_0. The simplest cases $w_0 = u_2 \exp(\gamma y)$, $w_0 = u_1 y + u_2 \exp(\gamma y)$ and $w_0 = (u_1 + u_2 y) \exp(\gamma y)$ lead to the examples that have been studied above (see the examples with initial conditions (78), (82) and (85), respectively). Qualitative analysis of (95) is highly non-trivial and will be done in another paper.

5. Conclusions

The methods described in section 2 lead to an open-ended set of possible exact solutions for systems of the form of (1)–(3); our presentation here has only scratched the surface. Larger values of m and n (see the generating system (4)) would lead to more complicated initial conditions and more difficult algebraic problems, but nothing that would be untractable with modern computational methods. This same approach could also be applied to similar one-dimensional Stefan-like problems.

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